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REMARKS ON KOBA-NIELSEN-OLESEN SCALING

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ABSTRACT It is shown that there is a second properly normalized KNO scaling function, $nP_n(n/\bar{n}) = \varphi(z)$, which has certain advantages in the analysis of KNO scaling. First, the nP_n are not influenced by the statistical and systematic uncertainties of $\bar{n}$ hence $\varphi(z)$ provides more selective power than the original KNO scaling function $\bar{n}P_n(n/\bar{n}) = \psi(z)$. Second, the new scaling function generates scale parameter $\sigma = 1$ since it depends only on the combination of z and the scale parameter of $\psi(z)$. An analysis of $\varphi(z)$ is given using e^+e^- annihilation data for charged particle multiplicity distributions.

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One of the most influential contributions to the analysis of multiplicity distributions was made more than 20 years ago by Koba, Nielsen and Olesen [1]. They put forward the hypothesis that at very high energies the probability distributions P_n for detecting n final state particles exhibit a scaling law of the form

$$P_n = \frac{1}{\bar{n}}\,\psi\left(\frac{n}{\bar{n}}\right). \tag{1}$$

That is to say, the $\bar{n}P_n$ measured at different energies (i.e. $\bar{n}$) scale to the universal curve ψ when plotted against the multiplicity n rescaled by the average multiplicity $\bar{n}$. This is the famous Koba-Nielsen-Olesen (KNO) scaling hypothesis. Defining the scaling variable $z \equiv n/\bar{n}$ the scaling function $\psi(z)$ must satisfy two normalization conditions:

$$\int_0^\infty \psi(z)\,dz = 1 \tag{2}$$

and

$$\int_0^\infty z\,\psi(z)\,dz = 1, \tag{3}$$

i.e. $\bar{z} = 1$. Obviously, the moments $\mathcal{M}_q$ of $\psi(z)$ are independent of collision energy if Eq. (1) is satisfied,

$$\begin{aligned}\mathcal{M}_q &\equiv \int_0^\infty z^q \psi(z)\,dz \\ &= \overline{z^q} = \overline{n^q}/\bar{n}^q = \text{const.}\end{aligned} \tag{4}$$

Concerning the KNO scaling form of P_n given by Eq. (1) let us make a further remark. For a mathematically specified distribution $F(x)$ σ is called a scale parameter if $F(x)$ has the form [2]

$$\frac{1}{\sigma}\,f\left(\frac{x}{\sigma}\right), \qquad \sigma > 0. \tag{5}$$

The parameter σ characterizes the dispersion, that how widely the values of x are spread around the typical value, reflecting thus the scale of the distribution. From Eq. (1) it is seen that in the KNO scaling form of P_n the average multiplicity serves as a scale parameter. This coincides with the fact that if KNO scaling holds the dispersion of P_n is measured, up to a constant of proportionality, by $\bar{n}$. The form Eq. (5) is obeyed by the scaling function too because the requirement $\bar{z} = 1$ usually generates a scale parameter $\sigma \neq 1$ for $\psi(z)$.

In the phenomenological analysis of multiplicity distributions one of the successfully applied probability laws is the negative binomial [3]. It has the form

$$P_n^{NBD} = \frac{\mathcal{P}(k,n)}{n!}\left(\frac{\bar{n}}{k}\right)^n \left(1+\frac{\bar{n}}{k}\right)^{-n-k} \tag{6}$$

where $\bar{n}$ and k are free parameters and

$$\mathcal{P}(k,n) = k\,(k+1)\ldots(k+n-1) = \frac{\Gamma(k+n)}{\Gamma(k)} \tag{7}$$

denotes the Pochhammer polynomial for (possibly non-integer) rising factorial powers [4]. Eq. (6) has a nice KNO scaling form: in the limit $\bar{n}/k \gg 1$ and $n/\bar{n}$ fixed

$$\bar{n}P_n^{\scriptscriptstyle NBD} \sim \psi_{\scriptscriptstyle G}(z) \tag{8}$$

where

$$\psi_{\scriptscriptstyle G}(z) = \mathcal{N}\cdot\theta^k z^{k-1}\exp(-\theta z) \tag{9}$$

is the gamma distribution with normalization constant $\mathcal{N} = \Gamma^{-1}(k)$, scale parameter θ and shape parameter k. $\psi_{\scriptscriptstyle G}(z)$ may have a variety of shapes, depending on the value of k, whereas θ reflects its scale, the degree of scatter of z around $\bar{z}$. The moments $\mathcal{M}_q$ of the gamma distribution are given by

$$\mathcal{M}_q = \theta^{-q}\,\mathcal{P}(k,q). \tag{10}$$

From the requirement $\mathcal{M}_1 = 1$ expressed by the normalization condition Eq. (3) we see that the scale parameter θ in Eq. (9) is enforced to satisfy

$$\theta = \mathcal{M}_1\big|_{\theta=1} = \mathcal{P}(k,1), \tag{11}$$

i.e. $\theta = k$. The scale parameter coincides with the shape parameter of the KNO scaling function. KNO scaling holds for energy independent k.

Since the work of Koba, Nielsen and Olesen testing the validity of the scaling hypothesis and the analysis of the scaling function $\psi(z)$ are of permanent interest in multiparticle phenomenology [5]. But there is a noteworthy fact that escaped attention: besides $\psi(z)$ there is a second properly normalized scaling function obeyed by the P_n. The requirement Eq. (3) for the first moment of $\psi(z)$ defines a KNO scaling function normalized to 1, $\varphi(z) \equiv z\,\psi(z)$. This yields the scaling law

$$nP_n = \varphi\!\left(\frac{n}{\bar{n}}\right) \tag{12}$$

for the multiplicity distributions, the nP_n should lie on the universal curve $\varphi(z)$ if Eq. (1) holds. The moments of the two KNO scaling functions are related by $\mathcal{M}_q^{(\varphi)} = \mathcal{M}_{q+1}^{(\psi)}$.

Investigating nP_n instead of $\bar{n}P_n$ has the obvious advantage that the statistical and systematic errors of $\bar{n}$ do not give contribution to the experimental uncertainty in the shape of the scaling function. Therefore $\varphi(z)$ can be more selective between various theoretical predictions than $\psi(z)$. Another noticeable property of $\varphi(z)$ arises in connection with Eq. (5). Since the KNO function $\psi(z)$ obeys this form, with σ constrained by Eq. (3), the scaling law in Eq. (12) can be written as

$$nP_n = \varphi\!\left(\frac{z}{\sigma}\right). \tag{13}$$

The nP_n depend only on the scaling variable z rescaled by the (energy independent) scale parameter σ of $\psi(z)$. Thus $\varphi(z/\sigma)$ provides a properly normalized KNO scaling function with scale parameter 1 and unconstrained $\mathcal{M}_1$. The gamma distribution exhibits the scaling law Eq. (13) with $\sigma = \theta^{-1}$. Using the new scaling variable

$$w = z\theta = \frac{z}{\overline{z^2} - 1} \tag{14}$$

(θ being restricted according to Eq. (11)) one arrives at the scaling function

$$\varphi(w) = \mathcal{N} \cdot w^k \exp(-w) \tag{15}$$

with $\mathcal{N} = \Gamma^{-1}(k)$. The nP_n should fall onto the above scaling curve if the $\bar{n}P_n$ follow a gamma distribution with constant shape parameter k.

It is of interest to find the family of distributions obeying the form Eq. (5). On the basis of the results of Ferguson [6] let us consider the following three different types of probability laws on $[0, \infty)$

- a) Lognormal distribution
- b) Pareto distribution, possibly inverse power distribution
- c) Generalized gamma or Stacy distribution.

If x is a random variable distributed according to a), b) and c) above, the random variable $y = \ln x$ will have a

- a') Normal distribution
- b') Exponential distribution, possibly negative or translated
- c') Generalized normal distribution

respectively. According to Ferguson these distributions play an important role in statistics since they are the only known ones that generate scale parameter families Eq. (5) of densities admitting complete sufficient statistics when samples of size ≥ 2 are available [6].

Among the six probability laws $a) - c')$ we turn our attention to the generalized gamma or Stacy distribution. The ordinary gamma, Eq. (9), and thus the exponential law b') appear as a special case. Moreover $a) - b)$, the lognormal and Pareto laws (the former one being very important in multiparticle physics [7]) can be obtained from c) in certain limits of the parameters [6]. We summarize some basic properties of this distribution to be used later. Further details can be found in the textbook of Johnson and Kotz [8]. The generalized gamma distribution has the form

$$\psi_{GG}(z) = \mathcal{N} \cdot \theta^k z^{\mu k - 1} \exp(-\theta z^\mu) \tag{16}$$

with scale parameter θ and shape parameter k as in Eq. (9) and normalization constant $\mathcal{N} = \mu\Gamma^{-1}(k)$. The additional parameter, the exponent μ, generalizes the gamma distribution in a simple and useful manner: Eq. (16) corresponds to a power with exponent $1/\mu$ of a gamma variable. This can be seen most clearly from the structure of the moments:

$$\mathcal{M}_q = \theta^{-q/\mu}\, \mathcal{P}(k, q/\mu), \tag{17}$$

the qth moment of the generalized gamma coincides with the (q/μ)th fractional moment of the ordinary gamma at fixed θ and k. To get properly normalized KNO function $\psi(z)$ having $\mathcal{M}_1 = 1$ the scale parameter θ must satisfy

$$\theta = \mathcal{M}_1\big|_{\theta=1} = \mathcal{P}^\mu(k, 1/\mu). \tag{18}$$

If the $\bar{n}P_n$ follow a generalized gamma distribution with energy independent shape parameter k and exponent μ the nP_n obey the scaling law expressed by Eq. (15) with the corresponding normalization constant $\mathcal{N}$ and scaling variable

$$w = z^\mu\theta = \frac{z^\mu}{\mu\left(\overline{z^{\mu+1}} - \overline{z^\mu}\right)} \tag{19}$$

as can be checked through eqs. (17) and (18). This gives back Eq. (14) for $\mu = 1$. It is worth noticing that the exponent μ is absorbed entirely into the rescaled variable w.

The above extension of the gamma distribution is not new in the physical literature. For a large class of nonlinear stochastic processes with pure multiplicative noise Eq. (16) arises as the stationary solution of the corresponding Fokker-Planck equation [9]. On the basis of the previous work the generalized gamma distribution was reviewed by Carruthers and Shih [5] pointing out that numerous theoretical KNO scaling functions are special cases. In its restricted form with $\mathcal{M}_1 = 1$ and θ given by Eq. (18) $\psi_{GG}(z)$ was rediscovered by Krasznovszky and Wagner [10]. In a series of papers they carried out a pioneering work in determining the magnitude of the exponent μ for a large amount of data available in various reactions, see [11] and references therein. Here we consider briefly the arguments made by Dokshitzer about the higher-order perturbative QCD effects on the shape of $\psi(z)$ in e^+e^- annihilations [12].

Over the past years much attention has been focused on the description of multiplicity distributions for hard processes in the framework of quantum chromodynamics (for a recent review see ref. [13]). Although the calculations predicted KNO scaling, which is indeed the case in e^+e^- annihilations, the scaling function proved to fall off exponentially in contradiction to the experimental results. According to observations [14] the tail of $\psi(z)$ decreases faster than exponential. In a recent attempt Dokshitzer was able to take into account more precisely the energy conservation in parton cascades by involving next-to-next-to-leading QCD effects in the determination of $\psi(z)$ and its moments [12]. The obtained KNO scaling function behaves according to

$$\psi(z) \propto \exp(-[Dz]^\mu) \tag{20}$$

for large z with calculable constant D and exponent

$$\mu = \frac{1}{1 - \gamma\left(\alpha_s(\mathcal{Q})\right)}. \tag{21}$$

In the above formula α_s is the QCD running coupling constant corresponding to the hardness scale $\mathcal{Q}$ and

$$\gamma\left(\alpha_s(\mathcal{Q})\right) = \frac{d \ln \bar{n}(\mathcal{Q})}{d \ln \mathcal{Q}} \tag{22}$$

is the multiplicity anomalous dimension responsible for the energy growth of the average multiplicity $\bar{n}$. In ref. [12] it was concluded that the better account of conservation laws drastically reduces the higher-order correlations and thus the width of $\psi(z)$. The exponent μ was found to be $\mu \approx 1.6$ which agrees with the results obtained in [11].

The comparison of eqs. (20) and (16) suggests that the generalized gamma distribution is well suited to characterize how the higher-order corrections of perturbative QCD affect the shape of $\psi(z)$. Let us take Eq. (9), the ordinary gamma with exponentially decreasing tail as a reference distribution. One can model the suppression of correlations by raising the gamma variable z to the power $1/\mu < 1$. With increasing z this procedure reduces more and more the contribution of the exponentially falling tail to the multiplicity moments $\overline{z^q}$. The original distribution will be squeezed, decreasing faster than exponential and having fractional multiplicity moments with rescaled rank q/μ in Eq. (10). The same method can be applied in the opposite case too: one can model possible enhancement of correlations by raising the gamma variable z to a power > 1. Recall that the exponent μ in Eq. (16) has already been determined for numerous reaction types [11]. This can provide valuable information on the degree to which the correlations are enhanced/suppressed relative to the KNO scaling form, Eq. (9), of the negative binomial distribution.

Having derived a new KNO-type scaling law for the multiplicity distributions, Eq. (12), it is tempting to check its validity on experimental data. We analysed the charged particle multiplicity distributions in e^+e^- annihilations from the ALEPH, AMY, ARGUS, DELPHI, HRS, L3, OPAL and TASSO collaborations in the c.m. energy range $\sqrt{s} = 9.4 - 91$ GeV [14]. Our aim was twofold. First, to determine how closely the nP_n from different energies follow a universal scaling curve $\varphi(z)$. Second, to measure how our decisive power changes, in terms of χ^2 statistics, if $\varphi(z)$ is fitted to theoretical predictions instead of $\psi(z)$.

Figure 1 displays the experimental data for nP_n plotted against $n/\bar{n}$ (since the points are densely populated no different symbols are used to represent different data sets). According to expectation, the nP_n follow a unique scaling curve in agreement with the observed KNO scaling behavior of the $\bar{n}P_n$ [14]. Another expected feature in connection with $\varphi(z)$ is the increase of χ^2 in fitting procedures. The nP_n are not influenced by the statistical and systematic errors of $\bar{n}$ hence the shape of $\varphi(z)$ is less uncertain than that of $\psi(z)$. Studying each data set separately we carried out fits both to nP_n and $\bar{n}P_n$ (by a theoretical $\varphi(z)$ and $\psi(z)$ respectively) in order to measure the significance of this effect in different circumstances. We used Eq. (16) for the theoretical $\psi(z)$ with normalization constant $2\mathcal{N}$ (because of the charged particle data), scale parameter θ given by

Eq. (18) and exponent μ fixed at the value $\mu = 1.6$. According to our results the increase in χ^2 is substantial: among the 11 analysed data sets it was found to be > 50 % for 9 distributions. In the fitted values of the shape parameter k no significant change was observed. The minimum and maximum increase in χ^2 correspond to the ALEPH and TASSO 34.8 GeV data respectively. In the former case the variation is only marginal:

$$\chi^2_\psi/\text{d.o.f.} = 11/23 \qquad \text{and} \qquad \chi^2_\varphi/\text{d.o.f.} = 13/23.$$

In the latter case the deviation is much more significant:

$$\chi^2_\psi/\text{d.o.f.} = 11/16 \qquad \text{and} \qquad \chi^2_\varphi/\text{d.o.f.} = 29/16.$$

An approximately 60 % increase in χ^2 is obtained when the fitting procedure was performed on the 11 distributions simultaneously using variable exponent μ. Again, no significant change was observed in the values of the fit parameters. These are $k = 6.06 \pm 0.34$ and $\mu = 1.29 \pm 0.03$, the exponent μ measuring the degree of deviation from an exponentially falling tail proved to be smaller than in refs. [11,12]. The scaling function $\varphi(z)$ corresponding to the fitted parameters is represented by the solid curve in Figure 1.

Let us summarize our main results. We have demonstrated that besides $\bar{n}P_n$ the more simple combination nP_n also scales to a universal curve in the variable $n/\bar{n}$ if KNO scaling holds valid. This somewhat unexpected behavior follows from the fact that the second normalization condition for $\psi(z)$, Eq. (3), defines a second properly normalized KNO scaling function, $\varphi(z) = z\,\psi(z)$. Thus Eq. (3) provides, on the one hand, certain restrictions: the first moment of $\psi(z)$ is constrained to be $\mathcal{M}_1 = 1$ and this usually generates a scale parameter $\sigma \neq 1$ for $\psi(z)$. On the other hand, Eq. (3) defines a KNO scaling function without the above restrictions. The qth moment of $\varphi(z)$ coincides with the $(q+1)$th moment of $\psi(z)$, further, $\varphi(z)$ generates scale parameter $\sigma = 1$ since it depends only on the combination of z and the scale parameter of $\psi(z)$. Perhaps the most profitable feature of the new scaling function lies in the fact that the statistical and systematic uncertainties of $\bar{n}$ are not involved by the nP_n. This makes $\varphi(z)$ more selective than $\psi(z)$. According to our experience gained in the analysis of 11 multiplicity distributions the increase in χ^2 statistics if the nP_n are fitted to a model prediction instead of $\bar{n}P_n$ exceeds 50 % for the majority of distributions. On the basis of the above features we conclude that the analysis of the scaling function $\varphi(z)$ may become a useful method in the exploration of physical processes that give rise to KNO scaling behavior for the multiplicity distributions.

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Figure Captions

Figure 1. The KNO scaling function $nP_n(n/\bar{n}) = \varphi(z)$ for charged particle multiplicity data in e^+e^- annihilations. The marked points represent ALEPH, AMY, ARGUS, DELPHI, HRS, L3, OPAL and TASSO data in the c.m. energy range $\sqrt{s} = 9.4 - 91$ GeV. The solid curve represents the theoretical $\varphi(z)$ corresponding to Eq. (16), see the text for details.

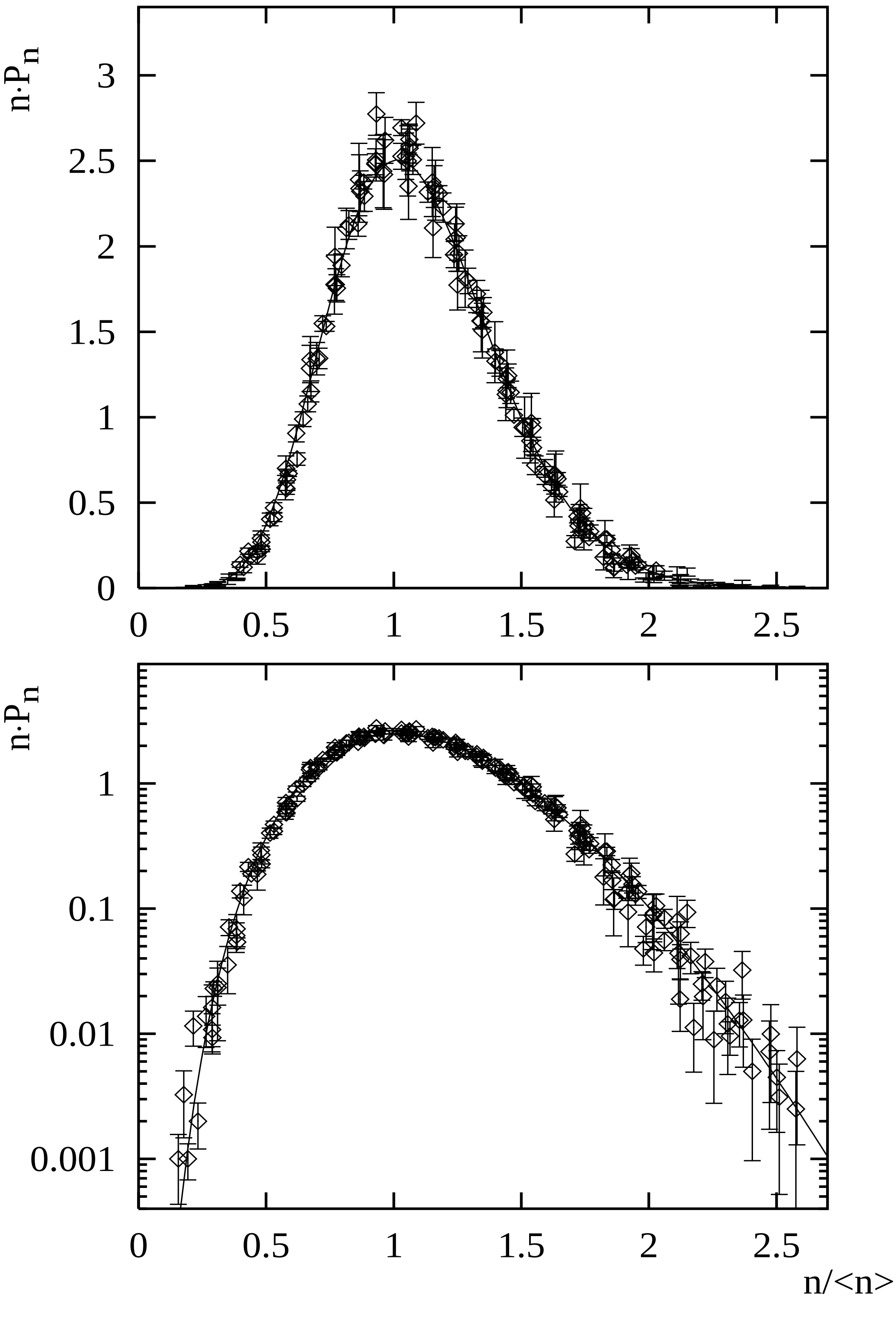

Figure 1